Recurrent sets of random walks and related topics

By Kenro ISHIDA

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§1. Introduction.

We are concerned with Markov processes (random walks) X_n, $n \geqq 0$ with state space

$$Z^d = \{ x \mid x \text{ has integer coordinates} \}$$

in d-dimensional space, $d \geqq 1$. The transition function is assumed to satisfy

$$0 \leqq P(x, y) = P(0, y-x)\,, \quad \sum_{x \in Z^d} P(0, x) = 1\,, \qquad x, y \in Z^d\,.$$

The probability of an event E concerning paths with initial point $X_0 = x \in Z^d$ will be denoted by $P_x\{E\}$. Our random walk can be denoted by

$$X_n = x + \sum_{i=1}^{n} Y_i = x + S_n\,, \qquad n \geqq 0 \tag{1}$$

where the Y_i are identically distributed independent random variables. As usual $P_n(x, y)$ will denote the n-th iterate of the transition function $P(x, y)$. Thus

$$P_0(x, y) = \delta(x, y)\,,$$
$$P_{n+1}(x, y) = \sum_{z \in Z^d} P(x, z) P_n(z, y)\,, \quad n \geqq 1\,.$$

Hence

$$P_n(x, y) = P_x\{X_n = y\} = P\{S_n = y - x\}\,, \quad n \geqq 0\,,$$

where $S_0 = 0$, $S_n = Y_1 + \cdots + Y_n$, as in equation (1). Thus $P\{A\}$ without a subscript will be used for events A defined in terms of the increments $Y_n = X_n - X_{n-1}$ of the random walk. We shall use $E\{\cdot\}$ to denote expectation corresponding to the measure $P\{\cdot\}$ and $E_x\{\cdot\}$ corresponding to $P_x\{\cdot\}$.

A random walk is called *aperiodic* if no proper subgroup of Z^d contains the set of x such that $P(0, x) > 0$. This is clearly no essential restriction, as the state space can always be redefined, if necessary, so

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as to make a random walk aperiodic.

Let

$$G(x, y) = \sum_{n=0}^{\infty} P_n(x, y)\,, \quad \mu = \sum_{x \in Z^d} x P(0, x)$$

and

$$m_i = \sum_{x \in Z^d} |x|^i P(0, x)\,, \qquad (i=1, 2, \cdots)\,,$$

here $|x|$ denotes the distance from x to the origin. A random walk is said to be *recurrent* or *transient* according as $G(0, 0) = \infty$ or $< \infty$. It is known (F. Spitzer [1] P. 83) that d-dimensional random walk is recurrent if and only if $d=1$ and $m_1 < \infty$, $\mu = 0$ or $d=2$ and $m_2 < \infty$, $\mu = 0$ and it is always transient when $d \geqq 3$.

We introduce further notations. For any set $A \subset Z^d$, we define

$$\begin{aligned}
T_A &= \min\{n \geqq 1 \mid X_n \in A\} & & \\
&= \infty & &\text{if } X_n \in Z^d - A \text{ for all } n \geqq 1\,, \\
H_A(x, y) &= P_x\{X_{T_A} = y\,;\, T_A < \infty\} & &\text{for } x \in Z^d - A\,, \quad y \in A\,, \\
&= \delta(x, y) & &\text{for } x \in A\,, \quad y \in A\,, \\
H_A(x) &= \sum_{y \in A} H_A(x, y) & &\text{for } x \in Z^d\,, \\
\Pi_A(x, y) &= P_x\{X_{T_A} = y\,;\, T_A < \infty\} & &\text{for } x \in A\,, \quad y \in A\,, \\
\Pi_A(x) &= \sum_{y \in A} \Pi_A(x, y) & &\text{for } x \in A\,.
\end{aligned}$$

We note that for $x \in Z^d$ $H_A(x)$ is the probability that the process X_n with $X_0 = x$ visits A at some time $0 \leqq n < \infty$ and that for $x \in A$ $\Pi_A(x)$ is the probability of revisiting A at some time $1 \leqq n < \infty$.

For any subset A of Z^d, $P_x\{X_n \in A$ infinitely often$\} = P\{A\}$ is independent of initial point $X_0 = x$ and can take only the values 0 or 1. A set A is said to be *recurrent* or *transient* according as $P\{A\} = 1$ or 0. For aperiodic transient random walk, F. Spitzer ([1] P. 300) proved that the set A is either recurrent in which case $H_A(x) = 1$ on Z^d or it is transient, and in that case $H_A(x) < 1$ for some $x \in Z^d$.

Given $A \subset Z^d$, a question arises as to this set A is recurrent or transient. It is known ([1] P. 275) that in recurrent random walk, any subset of Z^d is always recurrent and in arbitrary transient random walk, any finite set is transient. When $d=1$ and $m_1 < \infty$, $\mu \neq 0$, F. Spitzer [1] showed that if $\mu > 0$, then set A is recurrent if and only if $||A \cap \{x \mid x > 0$ and x is integer$\}|| = \infty$ and if $\mu < 0$, then set A is recurrent if and only if $||A \cap \{x \mid x<0$ and x is integer$\}|| = \infty$, where $||B||$ means the cardinal of set B. When $d \geqq 3$ and $m_2 < \infty$, $\mu = 0$, K. Itô-H. P. McKean [2], Spitzer [1] and others obtained the necessary and sufficient conditions for a set to be recurrent which is called "Wiener's test." However this test is too theoretical to decide whether the set is recurrent or not.

The main purpose of this paper is to give the practical criteria for the recurrent (transient) set in the case of $d \geqq 2$ and $\mu \neq 0$, $m_1 < \infty$. Suggested by the methods of R. Doney [3] and R. Bucy [4] for random walks of dimension $d \geqq 3$, in § 3 we give some results concerning the *regular* (*or irregular*) points of the Brownian motions.

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§ 2. Recurrent sets of the random walks.

Now we discuss random walks for $d \geqq 2$ and $\mu \neq 0$, $m_1 < \infty$. Hence μ is a vector. First of all, we give the sufficient condition for transient set applying the method which F. Spitzer used for the one-dimensional random walks. For simplicity we state and prove Theorem 1 for two-dimensional random walks. But this theorem holds also for higher dimensions without essential changes. We denote by $\widehat{KOK'}$ *sector* formed by the rays OK and OK' starting from the origin O.

Theorem 1. *For two-dimensional random walk with $\mu \neq 0$ and $m_1 < \infty$, if we can choose the angle $\theta = \angle KOP = \angle K'OP$ such that $||B \cap \widehat{KOK'}|| < \infty$, then the set B is transient, where OP is a half-line starting from the origin and passes through the point μ and $||B||$ means the cardinal of set B.*

Proof. Put $S_0 = 0$, $S_n = Y_1 + Y_2 + \cdots + Y_n$ as in equation (1). We can now denote $X_n = (X_n^{(1)}, X_n^{(2)})$ as vector, here $X_n^{(i)} (i=1, 2)$ are random variables taking integer, and μ can also be expressed by $(a, b) \neq (0, 0)$. By the law of large number, we obtain

$$P\left\{\lim \frac{S_n}{n} = \mu\right\} = P_0\left\{\lim \frac{X_n}{n} = \mu\right\}$$

$$= P_0\left\{\lim \frac{X_n^{(1)}}{n} = a \text{ and } \lim \frac{X_n^{(2)}}{n} = b\right\} = 1.$$

Set the events

$$E = \left\{\omega \middle| \lim \frac{X_n}{n} = \mu\right\}$$

and

$$E_{n,\varepsilon}=\{\omega|\ (n+i)(a-\varepsilon)<X_{n+i}^{(1)}<(n+i)(a+\varepsilon)\ ,$$
$$(n+i)(b-\varepsilon)<X_{n+i}^{(2)}<(n+i)(b+\varepsilon)\ ;\quad i=0,1,2,\cdots\}\ .$$

We easily get

$$E\subset\bigcup_n E_{n,\varepsilon}\quad\text{for arbitrary}\qquad \varepsilon>0\ .$$

Therefore

$$1=P_0\{E\}=P_0\{\bigcup_n E_{n,\varepsilon}\}\leqq\sum_n P_0\{E_{n,\varepsilon}\}\ .$$

Hence there is some n_0 such that $P_0\{E_{n_0,\varepsilon}\}>0$. Denote by F the event

$$\{\omega|\ (n_0+i)(a-\varepsilon)<X_i^{(1)}<(n_0+i)(a+\varepsilon)\ ,$$
$$(n_0+i)(b-\varepsilon)<X_i^{(2)}<(n_0+i)(b+\varepsilon)\ ;\quad i=0,1,2,\cdots\}\ .$$

It follows from the Markov property that

$$P_0\{E_{n_0,\varepsilon}\}=\sum_x P_0\{X_{n_0}=x\}P_x\{F\}\ .$$

And so there is some x_0 such that $P_{x_0}\{F\}>0$. Let the rays OK and OK' with slopes $(b+\varepsilon)/(a-\varepsilon)$ and $(b-\varepsilon)/(a+\varepsilon)$ form the sector $\widehat{KOK'}$ which contains the point μ.

Since $P_{x_0}\{F\}>0$, the random walk starting at x_0 is in the sector $\widehat{KOK'}$ successively with positive probability. Therefore, if the set B is in the exterior of $\widehat{KOK'}$, then the probability that the process X_n with $X_0=x_0$ hits the set B eventually is less than 1. Hence $H_A(x_0)<1$. This implies that the set B is transient. If only finite subset of the set B is in $\widehat{KOK'}$, we have the same conclusion because the union of a finite number of transient sets is also transient.

Remark. For the one-dimensional case, the converse of Theorem 1 holds, but for higher dimension, this is not true. Indeed, for higher dimension we know from ([1] P. 288) that

$$\lim_{|x|\to\infty}G(0,x)=0\ .$$

Therefore we can choose a sequence of lattice points $\{x_n\}$ on half-line OP with the slope of rational number such that

$$G(0,x_n)<\frac{1}{2^n}\ .$$

So we have

$$\sum_{n=1}^{\infty} G(0, x_n) < \infty .$$

Hence it follows from the Borel-Cantelli lemma that $\bigcup_{n=1}^{\infty}\{x_n\}$ is transient set.

Before giving a sufficient condition for recurrent set, we need the next proposition which is an independent property of dimension d and μ, m_i.

Proposition 1. *If set A is transient (recurrent), then for any x in Z^d, $A_x=\{a-x \mid a \in A\}$ is also transient (recurrent).*

Proof. When set A is transient, $H_A(x_0)<1$ for some x_0 in Z^d. We will show that $H_A(x_0)=H_{A_x}(y_0)<1$ for $y_0=x_0-x$. By the definition of $H_A(x, y)$, we get

$$\begin{aligned} H_A(x_0, y) &= P_{x_0}\{X_{T_A}=y\,;\, T_A<\infty\} \\ &= \sum_n P_{x_0}\{X_{T_A}=y\,;\, T_A=n\} . \end{aligned}$$

But

$$\begin{aligned} &P_{x_0}\{X_{T_A}=y\,;\, T_A=n\} \\ &\quad= \sum_{x_i \in Z^d-A} P(x_0, x_1)P(x_1, x_2)\cdots P(x_{n-1}, y) \\ &\quad= \sum_{x_i \in Z^d-A} P(x_0-x, x_1-x)P(x_1-x, x_2-x)\cdots P(x_{n-1}-x, y-x) . \end{aligned}$$

If $x_i \in Z^d-A$, then $x_i-x \in Z^d-A_x$, so we see that the last expression above is equal to

$$\begin{aligned} &\sum_{y_i \in Z^d-A_x} P(y_0, y_1)P(y_1, y_2)\cdots P(y_{n-1}, y-x) \\ &\quad= P_{y_0}\{X_{T_{A_x}}=y-x\,;\, T_{A_x}=n\} . \end{aligned}$$

i.e.,

$$H_A(x_0, y)=H_{A_x}(y_0, y-x) .$$

Hence we have

$$\begin{aligned} H_A(x_0) &= \sum_{y \in A} H_A(x_0, y) = \sum_{y \in A} H_{A_x}(y_0, y-x) \\ &= \sum_{y \in A_x} H_{A_x}(y_0, y) = H_{A_x}(y_0) . \end{aligned}$$

The next theorem is valid also for 3-dimension without essential changes. But unfortunately our method will not work for $d \geqq 4$.

Theorem 2. *Consider two-dimensional random walk with $\mu=$*

$(a, b) \neq (0, 0)$. *And suppose that the ratio b/a is rational. As in Theorem 1, let OP be a half-line starting from the origin and pass through the point μ. If there exists a parallel line l to half-line OP such that the next condition is satisfied, then the set A is recurrent set.*

We can divide the line l in the quadrant containing the points μ into infinitely many intervals such that each interval has the same length and it contains at least one point of A.

Proof. Let straight line p be orthogonal to half-line OP and meet at the origin O. We define $\bar{Z}'$ to be the set consisting of all points which are projected orthogonally to line p from Z^2. The set $\bar{Z}'$ is isomorphic to the set Z^1 of all the integers on the real line. Indeed this is possible from the fact that the slope of OP is rational. So we can define the new random walk with the state space $\bar{Z}'$ on line p. Let $\bar{x}_i$ be the orthogonal projection of $x_i^1, x_i^2, x_i^3, \cdots$ onto line p, where $x_i^n (n=1, 2 \cdots)$ is of course in Z^2. We define the transition function of new random walk as

$$\bar{P}(0, \bar{x}_i) = P(0, x_i^1) + P(0, x_i^2) + P(0, x_i^3) + \cdots .$$

It is easily seen that

$$\bar{\mu} = \sum \bar{x}_i P(0, \bar{x}_i) = 0 ,$$

$$\overline{m}_1 = \sum |\bar{x}_i| P(0, \bar{x}_i) < \infty .$$

Hence the new random walk over $\bar{Z}'$ can be regarded as the one-dimensional recurrent random walk. Therefore arbitrary point $\bar{x}_i$ in $\bar{Z}'$ is visited infinitely often with probability one. This implies that the original random walk in Z^2 hits infinitely often the set in Z^2 consisting of points which form the orthogonal projection $\bar{x}_i$. Combining Theorem 1 with Proposition 1, we easily get the conclusion of Theorem 2.

By the definition of recurrent set, we easily get the following

Theorem 3. *For arbitrary random walk X_n, $n \geqq 0$, set A is recurrent if and only if there exists a bounded, non-negative function $f(x)$ defined on Z^d with support in A such that*

$$P_x\{\sum_n f(X_n) = \infty\} = 1$$

for some x in Z^d.

Proof. The necessity of the condition is easily shown by putting $f(x) = \chi_A(x)$, here χ_A is the characteristic function of the set A.

Now let f be a bounded, non-negative function which has support in A. Define the random variable t_A to be the total time spent in the set A. If the set A is transient, then for any x in Z^d,

$$P_x\{t_A<\infty\}=1 .$$

Denote by T the event

$$\{\omega|t_A<\infty\} ,$$

then for any ω in T,

$$\sum_n f(X_n(\omega))\leqq(\sup f)\cdot t_A(\omega)<\infty .$$

Hence we get

$$P_x\{\sum_n f(X_n)<\infty\}=1 .$$

§ 3. Some results on the regular points of Brownian motion.

In this section we discuss the regular points of standard Brownian motion in $R^d, d\geqq 3$. We will begin with a short summary of relevant definitions and known facts concerning the regular points of Brownian motion.

Let $X=\{x(t)\,;t\geqq 0\}$ be a standard Brownian motion with state space $R^d, d\geqq 3$. We denote the first passage time for set A by

$$\sigma_A=\inf\{t>0 \mid x(t)\in A\} .$$

A point x is said to be *regular point* of B for the process X, if it holds

$$P_x\{\sigma_B=0\}=1 .$$

K. Itô-H. P. McKean [5] proved the following theorem which is known as the "Wiener's test." Let B be an analytic set and let x be its boundary point and set

$$B_k=\{y \mid 2^{-k}\leqq|y-x|\leqq 2^{-k+1}\}\cap B .$$

Then x is a regular point of B for the process X, if and only if

$$\sum_{k=1}^{\infty} 2^{k(d-2)}C(B_k)=\infty ,$$

where $C(B_k)$ means the Newtonian capacity of set B_k.

R. Doney [3] proved that for three-dimensional random walk with $\mu=0$ and $m_2<\infty$, there is no positive valued function f such that the condition of the form

$$\sum_{b \in B} f(|b|) = \infty$$

is necessary and sufficient for set B to be recurrent.

As for the Brownian motion in R^d, $d \geqq 3$, we can deduce the analogous result.

Theorem 4. *For any analytic set B and any boundary point x of B, there is no positive valued function f such that the condition of the form*

$$\int_B f(x-y)dy = \infty$$

is necessary and sufficient for point x to be a regular point of set B.

Proof. We assume $d=3$ for simplicity. Suppose that there exists a positive valued function f as stated above. Let A_n be $\{a \mid 2^{-n} \leqq |a| < 2^{-n+1}\}$, and $I(n, m, \alpha)$ be the rectangular block such that

$$\{a \mid 2^{-n}+m < a_1 \leqq 2^{-n}(1+\alpha)+m,\ 0 < a_2 \leqq 2^{-n}\alpha,\ -\alpha^{-(n+1)} < a_3 \leqq 2^{-(n+1)}\},$$

where a_1, a_2 and a_3 are the coordinates of a.

We use the following lemma due to R. Doney [3].

Lemma. If C_α^β is the capacity of a rectangular block of dimension $\beta \times \beta \times \beta\alpha$, then

$$a \frac{\alpha\beta}{\log \alpha} \leqq C_\alpha^\beta \leqq b \frac{\alpha\beta}{\log \alpha} \qquad \text{for} \quad \alpha > 1,$$

where a and b are positive constants.

Put $I = \bigcup I_r$, where $I_r = I(n_r, m_r, \alpha_r)$ and $\{n_r\}$ is an increasing sequence of real numbers, $\{m_r\}$ a sequence of non-negative real numbers and $\{\alpha_r\}$ a sequence of real numbers satisfying, for each $r \geqq 1$,

$$I_r \subset A_{n_r} \quad \text{and} \quad \alpha_r < 1.$$

Applying the above lemma, we get

$$a \frac{2^{-n_r}}{\log \alpha_r^{-1}} \leqq C(I_r) \leqq b \frac{2^{-n_r}}{\log \alpha_r^{-1}}.$$

Thus it follows from the Wiener's test that the point 0 is a regular point of set I if and only if

$$\sum_{r=1}^{\infty} (\log \alpha_r^{-1})^{-1} = \infty.$$

Put $F(n, m, \alpha) = \int_{I(n,m,\alpha)} f(a)da$, then the rest of the proof can be done in the same way as in R. Doney [3].

Finally we state a result which we are suggested by the paper of R. Bucy [4] concerning the random walk. Let $g(t, a, b)$ denote the transition density of Brownian motion.
i. e.,

$$g(t, a, b)=\frac{1}{(2\pi t)^{d/2}} e^{-|b-a|^2/2t}.$$

Further we introduce the Green function as follows,

$$G(a, b)=\int_0^\infty g(t, a, b)dt=\frac{\Gamma(d/2-1)}{(4\pi)^{d/2}}|a-b|^{2-d}.$$

Theorem 5. *Let A be a Borel subset of R^d, $d\geqq 3$. If there exists a non-negative function $f(\cdot)$ such that*

$$\int_{R^d} G(a, b)f(b)db=1 \quad \textit{for all} \quad a\in\bar{A},$$

then

$$P_x\{\textit{there exists } t_0 \textit{ such that } x_t\notin A \textit{ for any } t>t_0\}=1, \quad \textit{for all } x\in\bar{A}.$$

Here notation $\bar{A}$ means closure of A.

Proof. We assume that

$$P_x\{\text{for any } t, \text{ there exists } t_0>t \text{ such that } x_{t_0}\in A\}>0, \text{ for some } x\in\bar{A}.$$

As usual we can derive the following

$$\begin{aligned} E_x\left\{\int_0^\infty f(x_s)ds\right\} &=\int_0^\infty\int_{R^d} f(x_s)P_x(d\omega)ds \\ &=\int_0^\infty\int_{R^d} f(b)g(s, x, b)dbds \\ &=\int_{R^d} G(x, b)f(b)db \\ &=1. \end{aligned}$$

Define the random variable

$$\begin{aligned} T_t &=\inf\{s \mid s>t \quad \text{and} \quad x_s\in A\} \\ &=\infty \quad \text{if } x_s\notin A \quad \text{for all} \quad s>t. \end{aligned}$$

Then

$$1=E_x\left\{\int_0^\infty f(x_s)ds\right\}\geqq E_x\left\{\int_0^t f(x_s)ds\right\}+E_x\left\{\int_{T_t}^\infty f(x_s)ds;\ T_t<\infty\right\}.$$

Applying the strong Markov property and noting the fact that $x_{T_t} \in A$ as long as $T_t < \infty$, we get

$$E_x\left\{\int_{T_t}^{\infty} f(x_s)\,ds\,;\; T_t<\infty\right\} = E_x\left[E_{x_{T_t}}\left\{\int_0^{\infty} f(x_s)ds)\,;\; T_t<\infty\right\}\right]$$

$$= E_x\{1\,;\; T_t<\infty\}$$

$$= P_x\{T_t<\infty\}\,.$$

Set the event

$$K = \{\text{for any } t, \text{ there exists } t_0 > t \text{ such that } x_{t_0} \in A\}\,,$$

the $P_x\{T_t<\infty\}$ tends to $P_x\{K\}$ as $t \to \infty$. So the above inequality implies

$$1 \geqq 1 + P_x\{K\}\,.$$

Thus $0 \geqq P_x\{K\}$ which is a contradiction.

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Faculty of Science and Technology,
Science University of Tokyo